

**CLEARINGHOUSE
FOR FEDERAL SCIENTIFIC AND
TECHNICAL INFORMATION**

Hardcopy

\$ 3.00

Microfiche

\$.50

53

pp

72

/ **ARCHIVE COPY**

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
AEROELASTIC AND STRUCTURES RESEARCH LABORATORY
Technical Report 133-1

AD632744

**AN ACTUATOR DISC THEORY
THE SHED WAKE AT LOW TIP SPEED RATIOS**

by
J. P. Jones

FOR THE
Department of the Navy
Bureau of Naval Weapons
Contract NOw 65-139-d

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
AEROELASTIC AND STRUCTURES RESEARCH LABORATORY
Technical Report 133-1

AN ACTUATOR DISC THEORY FOR THE SHED WAKE AT
LOW TIP SPEED RATIOS

by
J.P. Jones

FOR THE
Department of the Navy
Bureau of Naval Weapons
Contract N0w 65-139-d

SUMMARY

A model of the flow through a helicopter rotor, suitable for estimating shed wake effects in hovering and at low tip speed ratios, is described. The method leads to the same result for the aerodynamic damping of blade bending oscillations as more complex theories. It is shown that the damping of small natural oscillations is to a first order unaffected by tip speed ratio. Interharmonic couplings, arising out of the convection of the shed vorticity parallel to the rotor disc, prove to be quite important, especially if resonance occurs. Also, in the presence of strong interharmonic coupling, the effective lift curve slope cannot be considered as a universal parameter, independent of the harmonic content of the blade motion.

TABLE OF CONTENTS

<u>Section</u>		<u>Page</u>
I.	INTRODUCTION	1
II.	DEVELOPMENT OF THE FLOW MODEL	4
III.	HOVERING THEORY	9
IV.	EXTENSION TO FORWARD FLIGHT	26
	4. 1 Forward Flight Model	26
	4. 2 Forward Flight Theory	31
V.	DISCUSSION	37
VI.	CONCLUSIONS: FURTHER DEVELOPMENTS	42
	REFERENCES	45

LIST OF FIGURES

- | | |
|----------|---------------------------------------|
| Figure 1 | Nomenclature for flow model |
| Figure 2 | Explanation for forward flight model |
| Figure 3 | Variation of F with tip speed ratio |

LIST OF SYMBOLS

$l_{\dot{z}}$	- aerodynamic damping derivative for flexural oscillations
$(\partial l_{\dot{z}} / \partial \mu)_{\mu \rightarrow 0}$	- rate of change of $l_{\dot{z}}$ with μ at very small tip-speed ratios
h	- wake spacing in two-dimensional model of hovering rotor
C	- lift deficiency function
σ	- solidity
λ	- inflow factor
ω	- angular velocity of the rotor
l	- representative radius
V	- inflow velocity = $\lambda \omega l$
u, v	- components of fluid velocity parallel to the x and y axes respectively
p	- pressure
χ	- stream function
ζ	- vorticity
u, v	- perturbation velocities
ζ_{pc}, ζ_{ps}	- cosine and sine components of vorticity shed at frequency p
α	- angle between blades
$p\epsilon$	- inter-blade phase angles
x_0	- distance between blades on linear actuator disc
U_s	- $ \omega_s $ propagation speed of disturbance along the discontinuity
Q	- Arbitrary \dagger . an arbitrary quantity
β	- phase angle
K_t	- circulation at the representative radius
b	- semi-chord at representative radius

C_L	- lift coefficient at representative radius
a	- lift curve slope at representative radius
α_o	- representative mean incidence
w	- induced velocity of the shed wake at representative radius
z	- upward displacement of representative radius
w'	- induced velocity at reference blade calculated from the actuator disc model
Δp	- pressure difference at reference blade
ψ	- ωt non-dimensional time and azimuthal blade position
Φ_{nc}, Φ_{ns}	- non-dimensional vorticity components
F	- $\chi/\omega l^2$ non-dimensional stream function
ϕ	- δ/ω
ξ	- x/l
η	- y/l
C_m, S_m, C_{2n}, S_{2n}	- arbitrary constants
δ	- phase angle between lift force and blade vibrational velocity
U_F	- forward flight speed
μ	- $U_F/\omega l$ tip-speed ratio
λ_o	- mean inflow coefficient

I. INTRODUCTION

One of the problems in the calculation of the response of rotor blades to the various forms of excitation which arise, is to assess the importance of the 'non-steady aerodynamic effects'. By this is almost universally meant the shedding of spanwise vortices from the trailing edge of a blade whose incidence is varying with time, although it really has a wider connotation. The shed wake has an effect on rotor blade oscillations which is an order of magnitude greater than its effect on the oscillations of fixed wings. Basically this is because the wake does not move much, relative to the disc, in the time taken for the next blade to occupy the position which one blade has at a given instant. Also the blade natural frequencies vary with rotational speed, so that in hovering flight the opportunity may occur for shed vorticity of the appropriate sign and phase to accumulate underneath, and close to, the rotor, with the result that the aerodynamic damping of flexural oscillations can become very small.^{(1), (2)} But in forward flight the shed vortices are convected away from the disc, and their axes are no longer parallel to the trailing edges of successive blades. They therefore cannot accumulate in quite the same way and their effect on the rotor must be expected to decrease. On the other hand it is important to know just how big is this decrease, for the blades are then in forced oscillation and should the damping remains small, amplitudes and stress fluctuations may be large.

But there are very considerable difficulties in the way of calculating non-steady aerodynamic effects in

forward flight. In particular convincing analytical, or 'closed form', solutions are scarce because tractable models of the flow are hard to devise. The principal reasons are the strong three-dimensional character of the flow - the blade loading varies rapidly spanwise and the wake is at least a skewed helix - and the fact that conditions are not steady in either of the two most convenient frames of reference. A subsidiary reason is the sheer complication involved, for the blades change their incidences at several frequencies simultaneously and there is some degree of coupling between frequencies. Despite these difficulties in recent years there have been considerable advances in the technique of calculating rotor blade airloads. Through the combined use of physical reasoning, experiment ⁽³⁾, ⁽⁴⁾ and large computers, ⁽⁵⁾, ⁽⁶⁾, ⁽⁷⁾ there has been a steady but striking improvement in our understanding of the problems involved. But of course the various theories are numerical in nature and involve large and complex calculations. As such they are not especially suitable for establishing trends - except in specific design cases. Moreover, at low tip speed ratios, which are of interest because fluctuating loads are large in that region, there are considerable problems in numerical analysis and some of the basic approximations are invalid. Piziali ⁽⁸⁾, for example, has shown that numerical calculations of the influence of the shed wake are most sensitive to the number of discrete vortices assumed. There is thus a strong incentive to find some form of 'closed' solution so that trends can be investigated and a new feel for the problem established.

This means reducing the real system to a model which is capable of precise and reasonably simple mathematical form-

ulation. It has already been pointed out that this is not easy to do and in fact the conceptual difficulties determine the approach, and colour the procedure, of this paper. But if the hovering case can be regarded as solved, then in principle at any rate it should be possible to deal with flight at very low tip-speed ratios as a short extension, or small perturbation, of hovering flight. Or conversely hovering may be regarded as the limit of forward flight at very low tip speed ratios. Thus we ought at least to be able to estimate the rate at which conditions change with tip-speed ratio in almost hovering flight. For instance if l_z is a measure of the aerodynamic damping of blade bending oscillations, then it should be possible to estimate the quantity $(\partial l_z / \partial \mu)_{\mu \rightarrow 0}$. The purpose of this paper is, therefore, to obtain and analyse a model of the flow through a rotor which provides a 'closed form' solution for the aerodynamic damping the vicinity of $\mu = 0$.

Now the intention is to regard forward flight as an extension of the hovering case, so that the first task is to establish a simple mathematical model for calculating the effects of the shed wake in hovering. This model must, of course, be sufficiently flexible for forward flight to be introduced without much additional analysis. To achieve this it is necessary to make some sweeping assumptions, if not in fact bare-faced distortions, but these do appear to be justified by the results - at least for the hovering case. We therefore begin by examining the important parameters, and reviewing previous closed solutions for the aerodynamic forces on rotor blades which are oscillating in hovering flight.

II. DEVELOPMENT OF THE FLOW MODEL

The principal parameters in the problem of the oscillating rotor blade in hovering are the number of cycles of the oscillation which occur in one revolution of the rotor (the frequency ratio), the pitch of the helical wake, the number of blades and the phase angle between the oscillations of the different blades. The frequency parameter (reduced frequency) is of secondary importance although this is only because it is small. Frequency ratio is important because it determines how the shed vorticity is distributed over the helical wake. A little reflection (or see (1), (2)), will show that if the frequency ratio is an integer the vorticity will have the same value at corresponding azimuth points in every turn of the helical wake. Thus the induced velocities of the individual turns are cumulative and it is this effect which causes the aerodynamic damping to be reduced. For an oscillation at a given frequency ratio the relative distribution of vorticity in the sheets from different blades depends upon the inter-blade phase angle. Thus even for oscillations at integral frequency ratios, the overall induced velocities will not be cumulative unless the blades are oscillating in the appropriate phase - which is a function of the frequency ratio and the number of blades. The final magnitude of the induced velocity must also depend upon how far the wake vortex sheets are from the blades i.e., upon the 'pitch' of the wake which in its turn depends upon the inflow velocity and the number of blades.

The basic difficulty in the determination of the forces on a rotor blade is the calculation of the induced

velocity, for the vortex sheets are spirals and their strength is not only initially unknown but varies both around and across the spiral. So far two models of the flow have been devised which avoid or reduce this difficulty. The first is the two-dimensional model of Jones ⁽²⁾ and Loewy ⁽¹⁾. They argue that since the aspect ratio of a rotor blade is high, and most of the load must be generated near the blade tips, then it is sufficient to treat the blades and their wakes as being plane and of infinite span. Thus the effects of the wake curvature and spanwise loading variations can be ignored and each turn of the spiral wake is represented by a separate, plane, vortex sheet below the disc and extending to infinity in front of and behind the blade. The sheets are spaced vertically at a distance h/b which is representative of the pitch of the helix, and the distribution of vorticity in the sheets depends only on the frequency ratio. One restriction on the usefulness of this model is that it can be applied to a multi-blade rotor, only if the inter-blade phase angle is assumed. For our present purpose the most important prediction of these theories is that, at an integral frequency ratio, the two dimensional lift curve slope for a vibrating blade section is

$$\left(\frac{\partial C_L}{\partial \alpha} \right)_{p=\text{integer}} = 2\pi \frac{h}{h + \pi} \quad (1)$$

i.e. the 'lift deficiency function' is $h / (h + \pi)$

The second model is due to Miller ⁽⁵⁾ who considers a circular rotor with an infinite number of blades. In this case the frequency ratio is restricted to integral values, but the influence of finite span and wake curvature is included. By considering a rotor which is uniformly loaded spanwise,

so that trailing vorticity leaves the blades only at the tips, the shed and trailing wakes are combined into a circular cylinder whose induced velocities can be calculated. It is assumed that the variations in the induced velocity across the chord of a blade can be neglected and lifting line theory is used to calculate the aerodynamic forces. The lift deficiency function C is shown to be

$$C = \frac{1}{1 + \frac{\sigma_1 \pi}{4 \lambda_0}} \quad (2)$$

so that the number of blades and the wake pitch appear through the ratio σ_1 / λ_0 . Here σ_1 is the solidity defined as (blade area/disc area) and λ_0 is the ratio of the mean inflow velocity to the tip speed.

Equation (2) agrees directly with Loewy's result for C if we put

$$\frac{h}{b} = \frac{4 \lambda_0}{\sigma_1} \quad (3)$$

The striking thing about these two theories is that they lead to essentially the same result. Yet neither theory is an extension of the other; the models used are in fact complementary since each contains features the other cannot reproduce. This agreement could be a meaningless chance, for it is possible that the common prediction for the lift deficiency function could be quite wrong. But measurements made by Shutler (reported in Jones⁽⁹⁾) have shown that the predictions of Loewy and Jones are in good agreement with experiment, even for the more difficult problem of torsional oscillations. Thus the evidence suggests

that the theoretical values for non-steady aerodynamic forces are insensitive to the details of the model on which the calculations are based. Therefore why should we not obtain a very simple model for the hovering case by combining the most (analytically) convenient features of the earlier solutions? If this alternative representation yields the same expression as equation (2), for the lift deficiency function then, assuming that it has the right flexibility, it should provide a very good basis for an extension to forward flight.

Now the simplest features of Miller's model are the use of lifting line theory and the assumption of an infinite number of blades. The greatest simplification introduced by Jones and Loewy is the idea that the flow is essentially two dimensional. If it is permissible to stretch these ideas to the limit in combination then we are led to consider a rotor which has a very large number of closely spaced blades each of very high aspect ratio. For the purposes of analysis this becomes a two-dimensional actuator disc, or surface of discontinuity, exactly similar to that employed in the most elementary momentum theories for the propeller. The difference is that since each part of the disc is to represent a blade which is oscillating or in some way changing its incidence with time, then we must postulate that vorticity is generated at every point of the discontinuity.

A great advantage of this model is that the analysis for it has already been worked out - or at least the basic procedure is established. Actuator disc theory has been widely applied to the calculation of flow through cascades. Sears ⁽¹⁰⁾ has used it to study the rotating stall but the

first real application to non-steady motion appears to be due to Whitehead⁽¹¹⁾ who used it to calculate the forces and moments on an aerofoil in an oscillating cascade. Although we shall use a different notation, and the lifting line theory will replace steady state cascade theory, and although we find it necessary to introduce the solidity to allow for the finite number of blades in a real rotor, the analysis given below is basically due to Whitehead. In fact much of the stimulus for the present work came from Whitehead's paper.

III. HOVERING THEORY

The actuator disc is shown in Figure 1 as a discontinuity surface lying along the axis of x , i.e., the (infinitesimal) chord lines of the airfoils which make up the disc lie along Ox . It is assumed that the system of co-ordinates is rotating with the rotor blades, so that these aerofoils are in a free stream of velocity ωl where ω is the angular velocity of the rotor and l is some representative radius. Thus, relative to the point at which it was shed, vorticity moves parallel to Ox with velocity ωl and downwards, away from the disc, at the mean inflow velocity $V = \lambda \omega l$. Above the actuator disc there is no vorticity and the air moves uniformly at the mean inflow velocity. The flow model of Figure 1 therefore describes the changing conditions on a 'reference' blade, represented by the lifting line at the origin O , as it travels around the rotor disc. In the analysis the actuator disc will be assumed to extend to infinity in both directions, i.e. ahead of and behind the reference blade. This is a great analytical convenience, which has a precedent in the work of Jones and Loewy, and can be justified in the following way. At any azimuth position the reference blade can be regarded as the median line - AB of Figure 2 - of a sector of the true disc. On either side of this median the other blades, and the vortices shed during previous revolutions, make an angle with OB which, due to the wake curvature, increases with the azimuth distance from OB . But over some sector the other blades and the shed vortices can be taken without significant error, to lie parallel to the reference blade, and the larger the representative radius l , the greater

the arc on either side of the median - and therefore the linear distance on the model - over which this is sensibly correct. If this arc length is many chords in length, i.e. of the order of the blade radius, then the linear model may be safely allowed to extend to infinity since the remote parts, which admittedly are grossly unrepresentative, will make no significant contribution to conditions at the origin.

The equations governing the flow in the x-y plane are those of two-dimensional, inviscid rotational fluid motion viz, the equation of continuity

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0 \quad (4)$$

and the dynamical equations of motion

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + v_1 \frac{\partial u_1}{\partial y} = -\frac{1}{\rho} \frac{\partial p_1}{\partial x} \quad (5)$$

$$\frac{\partial v_1}{\partial t} + u_1 \frac{\partial v_1}{\partial x} + v_1 \frac{\partial v_1}{\partial y} = -\frac{1}{\rho} \frac{\partial p_1}{\partial y} \quad (6)$$

where u_1, v_1 are the fluid velocity components parallel to the x and y axes respectively, and p_1 is the pressure.

From (4) it follows that there is a stream function χ such that

$$u_1 = \frac{\partial \chi}{\partial y}, \quad v_1 = -\frac{\partial \chi}{\partial x} \quad (7)$$

The pressure p_1 may be eliminated from (5), (6) to give the vorticity equation

$$\frac{\partial \zeta}{\partial t} + u_1 \frac{\partial \zeta}{\partial x} + v_1 \frac{\partial \zeta}{\partial y} = 0 \quad (8)$$

where
$$\zeta = \frac{\partial v_1}{\partial x} - \frac{\partial u_1}{\partial y} \quad (9)$$

From (4), (7), (9) it follows that

$$\frac{\partial^2 \chi}{\partial x^2} + \frac{\partial^2 \chi}{\partial y^2} = -\zeta \quad (10)$$

We now assume that the velocities caused by the shed vorticity and the blade motion are small perturbations of the velocities ωl and $\lambda \omega l$ i.e. we put

$$u_1 = \omega l + u \quad (11)$$

$$v_1 = \lambda \omega l + v \quad (12)$$

so that, on neglecting higher powers of the perturbation velocities, the vorticity equation becomes

$$\frac{\partial \zeta}{\partial t} + \omega l \frac{\partial \zeta}{\partial x} + \lambda \omega l \frac{\partial \zeta}{\partial y} \quad (13)$$

where
$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \quad \text{and} \quad u = \frac{\partial \chi}{\partial y}, \quad v = -\frac{\partial \chi}{\partial x}$$

For the mean, i.e. unperturbed, flow the equation of continuity is satisfied by making $\nabla = \lambda \omega l$ the same above and below the axis. But since continuity across the actuator disc must also be satisfied by the oscillatory velocity components, it is necessary to impose the additional condition that

$$v_{01} = v_{02} \quad (14)$$

In (14) the first suffix (zero) indicates that the perturbation velocity is evaluated at the disc ($y=0$), and in the second suffix, 1 refers to the region above the disc and 2 to the region below. The distribution of vorticity in the lower half of the x-y plane - region 2 - depends

upon the frequency ratio and the inter-blade phase angle. In the two-dimensional model with discrete wakes these parameters can be regarded as separate, but in the present model they cannot be distinguished because the number of blades is supposedly infinite. Moreover the concept of a phase angle is not so appropriate to the linear actuator disc as it is to the real circular rotor. A more suitable idea is to assume that a wave of disturbance is propagating along the surface of discontinuity. For suppose the vorticity $\zeta(0,0,t)$ shed by the reference blade to be of the form

$$\zeta(0,0,t) = \zeta_{pc} \cos pt + \zeta_{ps} \sin pt \quad (15)$$

Then the vorticity shed by a blade spaced an angle α from the reference blade will be

$$\zeta = \zeta_{pc} \cos p(t - \epsilon) + \zeta_{ps} \sin p(t - \epsilon) \quad (16)$$

where $p\epsilon$ is a phase angle - or more precisely ϵ is a time delay - which in general will be a function of α and p . But on the linear actuator disc α becomes a distance, x_0 say, so that the time delay $\epsilon(\alpha, p)$ must be transformed into a dependence upon x_0 . This is easily done if we put $\epsilon = x_0 / U_s$ where $U = \omega_s l$ is a characteristic speed which depends upon the frequency ratio and the inter-blade phase angle. Then

$$\zeta(x_0, 0, t) = \zeta_{pc} \cos p\left(t - \frac{x_0}{U_s}\right) + \zeta_{ps} \sin p\left(t - \frac{x_0}{U_s}\right) \quad (17)$$

which repeats itself at intervals of time of $\frac{2\pi}{p}$ and intervals of x_0 of $\frac{2\pi U_s}{p}$ i.e. it is the equation of a wave travelling along Ox with speed U_s .

The vorticity $\zeta(x,y,t)$ at an arbitrary point $R(x,y)$ can be related to the way in which vorticity is shed from the disc, since (see Figure 1), the vorticity at R left x_0 at an earlier time $t - \frac{Y}{V}$.

Thus

$$\zeta(x,y,t) = \zeta_{pe} \cos p \left(t - \frac{Y}{V} - \frac{x_0}{V\omega_s} \right) + \zeta_{ps} \sin p \left(t - \frac{Y}{V} - \frac{x_0}{V\omega_s} \right) \quad (18)$$

But

$$x_0 = x - \omega y / V \quad (19)$$

Hence

$$\zeta(x,y,t) = \zeta_{pe} \cos p \left(t - \frac{x}{V\omega_s} - \frac{Y}{V} \left(1 - \frac{\omega}{\omega_s} \right) \right) + \zeta_{ps} \sin p \left(t - \frac{x}{V\omega_s} - \frac{Y}{V} \left(1 - \frac{\omega}{\omega_s} \right) \right) \quad (20)$$

Equation (20) shows that the vorticity distribution in the direction depends upon the value of ω/ω_s i.e. upon the absolute propagation speed of the wave which determines the inter-blade phase angle. We can, of course, proceed with the analysis without specifying the ratio $\frac{\omega}{\omega_s}$ but there is a considerable simplification if at this stage we introduce the value appropriate to the blade oscillations which occur in forward flight. In forward flight, conditions at each azimuth are determined only the cyclic variations in relative wind speed; they are certainly independent of whichever blade happens to be passing through that azimuth. Thus the various harmonics of the blade motion must adjust their phase so that at each azimuth conditions on any one blade are the same as those which obtain on any other blade at that azimuth. To see how this is achieved consider any two blades spaced an angle Δ apart in azimuth, and let some quantity Q (life, blade motion, etc.) associated with the leading blade be of the form $Q_1 = H \cos pt$. Then the same quantity for the second blade must be of the form $Q_2 = H \cos(pt - \beta)$ where β is the

phase angle which we must determine. But when $t = t + \alpha/\omega$ the second blade will replace the first at any chosen azimuth, and the variable Q_2 is now equal to $A \cos(\omega t - \beta + p\alpha/\omega)$. Thus $\beta = p\alpha/\omega$ and $Q_2 = A \cos p(t - \alpha/\omega)$. The angle α is essentially a distance-measured around the azimuth, so that the distribution of inter-blade phase angle is equivalent to a wave travelling around the disc at the rotational speed. Alternatively, we note that for the sort of oscillations which occur in forward flight, an observer outside the rotor disc, but moving with the velocity of the hub, sees that any variable such as Q varies from azimuth to azimuth, but that its value at a given azimuth is independent of time. To this observer the distribution of Q around the disc takes the form of a stationary wave, and oscillations of this type can be represented on the actuator disc model by putting $\omega = \omega_s$. This substitution considerably simplifies the analysis since it makes the vorticity distribution independent of γ . At the same time it illustrates the connection between interblade phase angle and frequency ratio, since the same independence of γ is observed if the blades of a hovering rotor are oscillating, in the appropriate phase relation, at an integral number of cycles per revolution. The method can, of course, accommodate arbitrary combinations of frequency ratios and phase angles but the proper value of U_s must then be worked out separately for each case.

The usual application of actuator disc theory is not to the calculation of the forces on individual blades, but to the determination of the forces exerted by the rotor as a whole. (The reactions of the fluid which balance these forces can be calculated very easily by applying the momentum and energy theorems of fluid mechanics.) Miller ⁽⁵⁾

has in fact used this approach to show that the lift deficiency function for the response of a complete rotor to changes in collective pitch is given by (2). This must be one of the first applications of simple momentum theory to non-steady motion, (although its use for quasi-steady problems has been justified (12)), and it provides an important extension to classical propeller theory.

But when the incidence changes are periodic this approach is not entirely adequate, since only those frequencies which are simultaneously integral multiples of the rotational speed, and of the number of blades, can contribute to the resultant force on the rotor. It is then necessary to use a theory which takes into account the conditions experienced by individual blades as they rotate. The simplest of this type, and the one we shall adopt here, is the lifting line theory. The validity of this theory is questionable, particularly for large frequency parameters or if the 'in-phase' component of the aerodynamic force is required, but these restrictions are quite compatible with the other assumptions which have been made.

Let the circulation about a blade at the representative radius be K ; then the lift per unit span L at that radius is

$$L = \rho \omega K \quad (21)$$

But
$$K = b C_L \omega \quad (22)$$

Where
$$C_L = a \left[\alpha_0 - \frac{\dot{z} + W}{\omega l} \right] \quad (23)$$

Here α_0 is the mean incidence, \dot{z} is the upward displacement

of the blade section and w is the induced velocity - measured positive downwards - of the shed vortex system - all evaluated at the representative radius. Considering only the time dependent part of $K (= K_t)$

$$K_t = -2b(\dot{z} + w) \quad (24)$$

Note that (24) does not contain the relative wind speed, so that the same relation between K_t and \dot{z} holds in forward flight. Now in an aerodynamic calculation \dot{z} must be regarded as known since it is just as much a function of the structural, as of the aerodynamic, properties of the blade. Thus to calculate the lift as a function of \dot{z} it is necessary to determine the relation between K_t and w . But the intention is to use actuator disc theory to evaluate w i.e., w is calculated on the assumption that the number of blades is infinite, whereas K_t is the circulation around a single blade with a finite level of loading. We have therefore to assume a relationship between the circulation about a finite blade and the distribution of vorticity in the surface of actuator disc.

Miller ⁽⁵⁾ overcomes this difficulty by in effect assuming the blade loading to be distributed over part of the circular disc and the shed vorticity to be distributed over the corresponding part of the wake. This approach is not, however, suitable to the linear model in which the frequencies and phase angles are arbitrary. Instead we assume the loading at a point on the actuator disc to be the same as the mean density of loading $(= \frac{2\rho\omega}{b} K_t)$ on an actual blade at the corresponding azimuth. This of course gives values for the induced velocity w which are far too high, because

the rotor is being assumed to be much too 'solid'. But since the system is linear the increase of scale is the only distortion introduced i.e., no phase shifts or additional frequencies are created by assuming every point of the actuator disc to be much more heavily loaded than it really is. Therefore, when we come to calculate the induced velocity at an actual blade, it is merely necessary to multiply the actuator disc value for this velocity - say - by some factor which allows for the finite number of blades in the real rotor. The obvious factor to use is the solidity

i.e.
$$W = \sigma W' \quad (25)$$

where σ is the solidity appropriate to the linear actuator disc theory, viz. $\frac{Nb}{\pi r}$ which is the fraction of the representative circumference occupied by blade cross-sections.

To calculate the loading on the reference blade we have to determine the pressure difference across it, and then relate this pressure difference to the circulation about the blade and hence to the vorticity which is shed into the wake.

From the linearised dynamical equations

$$P_1 + \rho \omega u = -\rho \int \left(\frac{\partial u}{\partial x} - \lambda \omega \right) dx \quad (26)$$

so that the pressure difference ΔP between the lower and the upper surfaces at a point on the reference blade is

$$\Delta P = -\rho \omega (u_{02} - u_{01}) - \rho \left(\frac{\partial^2}{\partial x^2} + \lambda \omega \frac{\partial}{\partial y} \right) \int (u_{02} - u_{01}) dx \quad (27)$$

Now the circulation around the reference blade is

$$K_t = \int_{chord} (u_{02} - u_{01}) dx$$

so that in accordance with the lifting line theory the force per unit span on it is

$$L = \rho \omega l \int_{\text{chord}} (u_{02} - u_{01}) dx \quad (28)$$

But L is also equal to the integral of the pressure difference across the chord

$$\text{i.e. } L = \rho \omega l \int_{\text{chord}} (u_{02} - u_{01}) dx - \rho \left(\frac{\partial}{\partial t} + \lambda \omega l \frac{\partial}{\partial y} \right) \int dx \int_{\text{chord}} (u_{02} - u_{01}) dx \quad (29)$$

Reversing the order of integration in the last term of (29)

$$L = \rho \omega l \int_{\text{chord}} (u_{02} - u_{01}) dx - \rho \int dx \left(\frac{\partial}{\partial t} + \lambda \omega l \frac{\partial}{\partial y} \right) \int_{\text{chord}} (u_{02} - u_{01}) dx \quad (30)$$

On comparing (28) and (30) it follows that we must have

$$\frac{\partial K_t}{\partial t} + \lambda \omega l \frac{\partial K_t}{\partial y} = 0 \quad (31)$$

But this is precisely true, since (31) merely expresses the fact that the changes in the circulation about a blade are convected away, with velocity \mathbf{V} normal to the disc, as vortices of opposite sign, and this idea has already been used to establish the distribution of vorticity in the wake - see equations (19), (20).

An alternative form of (31) is

$$\frac{\partial}{\partial t} (u_{02} - u_{01}) = \lambda \omega l \xi. \quad (32)$$

where ξ is the vorticity shed at the origin i.e., at the reference blade.

Whitehead⁽¹¹⁾ arrives at the same result by imposing the condition that the difference in total head across the actuator disc is independent of X . Since

$P + \rho \omega |u|$ is the linearised total head this assumption can be stated as

$$\frac{\partial}{\partial x} (P + \rho \omega |u|) = 0 \quad (33)$$

and (32) follows at once from (26).

From (21) and (24) lifting line theory gives for the fluctuating part of the lift

$$L = -\rho a b \omega |(\dot{z} + w)| \quad (34)$$

where $w = \nabla w'$ and w' is the normal component of the induced velocity at the reference blade i.e.

$$w' = \gamma_{01} = \gamma_{02} \quad \text{from (14)} \quad (35)$$

Substituting (35) in (34), and equating the two expressions for the lift we have

$$\int_{\text{chord}} (u_{02} - u_{01}) dx = -a b (\dot{z} + \nabla \gamma_{01}) \quad (36)$$

We next assume that the integral across the chord is equal to the value of $u_{02} - u_{01}$ at the reference blade multiplied by the chord, i.e., it is assumed that the pressure difference is constant across a chord length. This seems to be a legitimate assumption provided that the frequency parameter is small.

$$\text{Hence } u_{02} - u_{01} = -(a/2) (\dot{z} + \nabla \gamma_{01}) \quad (37)$$

So far we have three equations, (14), (32), (37)

to be solved, in terms of the velocity \dot{Z} , for the four unknowns $u_a, u_{az}, v_{a1}, v_{a2}$. A fourth relation can be obtained from (10).

To solve this partial differential equation the variables are first put into non-dimensional form by means of the transformations

$$\begin{aligned} \psi &= \omega t & x &= l\xi & \chi &= \omega l^2 F & S_{ps} &= \omega \Phi_{ns} \\ p/\omega &= \eta & y &= l\eta & S &= \omega \phi & S_{pc} &= \omega \Phi_{nc} \end{aligned} \quad (38)$$

Then substituting equation (20) for the vorticity in (10) we obtain

$$\frac{\partial^2 F}{\partial \eta^2} + \frac{\partial^2 F}{\partial \xi^2} = -\frac{\Phi_{nc}}{l} \cos n(\psi - \xi) - \Phi_{ns} \sin n(\psi - \xi) \quad (39)$$

The solution for F consists of a complementary function plus a particular integral. Above the disc, where there is no vorticity, the particular integral is zero. Below the disc the vorticity causes the particular integral of (39) to vary periodically with ξ and the complementary functions above and below the disc must possess this same periodicity. This if F_1 is the non-dimensional stream function in the upper half of the plane

$$F_1 = \{C_{1n} \cos n(\psi - \xi) + S_{1n} \sin n(\psi - \xi)\} \exp(\pm n\eta) \quad (40)$$

where the C_{1n}, S_{1n} are arbitrary constants which are to be determined. Either the positive or negative exponent is possible but the only permissible solution is one which remains finite as η tends to infinity. Therefore, since η is positive above the disc we take

$$F_1 = \{C_n \cos n(\psi - \xi) + S_n \sin n(\psi - \xi)\} \exp(-n\eta) \quad (41)$$

The complementary function for the lower half of the ξ, η plane has a similar form, but the positive exponent must be used in that region. Thus the complete non-dimensional stream function below the disc is

$$F_2 = \{C_{2n} \cos n(\psi - \xi) + S_{2n} \sin n(\psi - \xi)\} \exp(n\eta) + \frac{\Phi_{nc}}{n^2} \cos n(\psi - \xi) + \frac{\Phi_{ns}}{n^2} \sin n(\psi - \xi) \quad (42)$$

where C_{2n}, S_{2n} are arbitrary as before. (42) is easily verified by direct substitution in (39).

Now the normal components of the perturbation velocity must be continuous across the actuator disc, i.e., $V_{o1} = V_{o2}$ which becomes

$$\left(\frac{\partial F_1}{\partial \xi}\right)_0 = \left(\frac{\partial F_2}{\partial \xi}\right)_0 \quad (43)$$

where the notation $()_0$ means that the derivative is evaluated along the x-axis.

Making use of the expressions (41), (42) for F_1 and F_2 and equating the coefficients of $\sin n(\psi - \xi)$, $\cos n(\psi - \xi)$ separately to zero, equation (43) becomes

$$C_{1n} = C_{2n} + \frac{\Phi_{nc}}{n^2} \quad (44)$$

$$S_{1n} = S_{2n} + \frac{\Phi_{ns}}{n^2} \quad (45)$$

In non-dimensional form, the relation (32) between the bound and shed vortices becomes

$$\frac{\partial}{\partial \eta} \left[\frac{\partial F_2}{\partial \eta} - \frac{\partial F_1}{\partial \eta} \right]_0 = \lambda \phi \quad (46)$$

which, on substituting for F_1, F_2 and ϕ and equating coefficients of $\sin n(\psi - \xi), \cos n(\psi - \xi)$, reduces to

$$C_{2n} + C_{1n} = \lambda \Phi_{ns} / n^2 \quad (47)$$

$$S_{2n} + S_{1n} = \lambda \Phi_{nc} / n^2 \quad (48)$$

Finally the non-dimensional form of (37), which is a form of the relation between the bound vorticity and the circulation around the blades becomes

$$\left(\frac{\partial F_2}{\partial \eta} - \frac{\partial F_1}{\partial \eta} \right)_0 = -\frac{\partial}{\partial \xi} \left\{ \frac{\dot{z}}{1} - \nabla \left(\frac{\partial F_1}{\partial \xi} \right)_0 \right\} \quad (49)$$

To solve (49) we have to assume that \dot{z} is periodic with frequency ratio n .

Hence we put

$$z = | \{ z_{nc} \cos n\psi + z_{ns} \sin n\psi \} \quad (50)$$

Substituting for F_1 and F_2 in (49), and equating the coefficients of $\cos n\psi, \sin n\psi$ we get

$$C_{2n} + C_{1n} = \frac{\partial}{\partial \xi} \{ z_{ns} + \nabla S_{1n} \} \quad (51)$$

$$S_{2n} + S_{1n} = -\frac{\partial}{\partial \xi} \{ z_{nc} + \nabla C_{1n} \} \quad (52)$$

Thus, in terms of the blade vibrational velocity, we have six equations, (44), (45), (47), (48), (51), (52) for the six unknowns Φ_{nc} , Φ_{ns} , C_m , S_m , C_{2m} , S_{2m} . Eliminating the arbitrary constants we obtain

$$\frac{\Phi_{ns}}{\eta^2} \left\{ 1 + \frac{4\lambda}{2\sigma} \right\} + \lambda \frac{\Phi_{nc}}{\eta^2} = -\frac{2}{\eta\sigma} \eta z_{ns} \quad (53)$$

$$-\frac{\lambda \Phi_{ns}}{\eta^2} + \frac{\Phi_{nc}}{\eta^2} \left\{ 1 + \frac{4\lambda}{2\sigma} \right\} = -\frac{2}{\eta\sigma} \eta z_{nc} \quad (54)$$

The fluctuating lift per unit span on the reference blade is

$$L = \rho \omega l c (u_{02} - u_{01}) = \rho \omega^2 l^2 \lambda c \int \phi_0 d\psi \quad (55)$$

$$= -\rho \omega^2 l^2 \lambda c \left\{ \frac{\Phi_{nc}}{\eta} \sin \eta \psi - \frac{\Phi_{ns}}{\eta} \cos \eta \psi \right\} \quad (56)$$

Solving (53), (54) for Φ_{nc} , Φ_{ns} and substituting in (56) we have finally

$$\frac{L}{\rho \omega^2 l^2 c} = -\frac{2\lambda}{\sigma} \left[\frac{1}{\lambda^2 + \left(1 + \frac{4\lambda}{2\sigma}\right)^2} \right]^{1/2} \left\{ \begin{array}{l} -z_{nc} \sin m(\psi + \delta) \\ + z_{ns} \cos m(\psi + \delta) \end{array} \right\} \quad (57)$$

where

$$\tan \delta = \frac{\lambda/m}{\left[1 + \frac{4\lambda}{2\sigma}\right]} \quad (58)$$

Now the expression

$$m [-z_{nc} \sin m\psi + z_{ns} \cos m\psi]$$

is the non-dimensional form of the blade vibrational velocity, so that the lift force is out-of-phase with this velocity by an amount which depends upon λ . But λ is usually very small, and if we neglect terms in λ^2 and ignore δ a first approximation to (57) is

$$\frac{L}{\rho \omega^2 |z_c|^2} = \frac{-\frac{2}{\sigma} \left(\frac{2\lambda}{\sigma} \right)}{\frac{2}{\sigma} + \left(\frac{2\lambda}{\sigma} \right)} m \left[-z_{mc} \sin m\psi + z_{ms} \cos m\psi \right] \quad (59)$$

Thus the lift primarily is proportional, but of opposite sign, to the vibrational velocity i.e., it exerts a damping force on the oscillation. This approximate result can also be derived more quickly by eliminating the unimportant cross-couplings $\lambda \Phi_{mc}$ from (53), and $\lambda \Phi_{ms}$ from (54).

The more complete formula (57) shows that there is in fact a component of force in phase with the displacement i.e., the aerodynamic forces also supply a stiffness. However, this term can be ignored without difficulty since it will make a negligible change to the total stiffness of a blade, and the lifting line theory is in any case not accurate for the 'in-phase' components.

The term $\frac{(2\lambda/\sigma)}{(1 + \frac{4\lambda}{\sigma})}$ is the lift deficiency function for the oscillations. Observing that $\sigma = \frac{1}{2}\sigma$, this reduces to the same formula for C as was obtained by Loewy⁽¹⁾ and Miller⁽⁵⁾.

We have thus developed a method of calculating the lift deficiency function which gives the same results as, and is easier in use than, earlier theories. The next step is to extend the calculation to forward flight and this means stretching the model to its utmost, for we are restricted to dealing with time-varying problems in two-dimensions. Some indulgence has therefore to be granted when reading the following sections, but it is hoped that the approach will

be taken seriously for the main purpose is not so much to provide numerical results, as to establish a new vehicle, or framework, of understanding for the forward flight problem. Despite its limitations the theory bears many of the hallmarks of a forward flight analysis - complications, the existence of many frequencies, differential equations with periodic coefficients, inter-harmonic couplings, etc., some of which have not been considered before. Furthermore a perturbation procedure is used - not for the first time in rotor analysis - but in an aerodynamic, as distinct from a dynamic, calculation. The use of these techniques as a means of simplifying rotor analysis has recently been advocated by Young⁽¹³⁾ and it is hoped that this example of their use will provide a basis for future development. For example it may be that the best approach to the calculation of rotor loadings and noise at low tip speed ratios is to expand the wake geometry in powers of the tip-speed ratio - as well as other parameters.

IV. EXTENSION TO FORWARD FLIGHT

4. 1 Forward Flight Model

The principal differences between hovering and forward flight are that the inflow velocity and the tangential component of the relative wind speed vary periodically as a blade rotates, and that the blades are in forced oscillation. The relative wind speed variation has a period of once per revolution, but the induced velocity and the blade motion may have significant content at all frequencies up to ten per revolution.

The induced velocity variations are easily introduced into the model of Section 2, because the lifting line theory does not recognise a difference between the velocity of the air and the velocity of a blade normal to the disc. Hence we merely replace the velocities $\dot{\mathbf{z}} = |\partial \mathbf{z}_n / \partial \psi|$ of the hovering analysis by the relative air velocities $\partial \mathbf{z}_n / \partial \psi + \sum_n \{ \lambda_n(\psi) \}$ where $\lambda_n(\psi)$ is the (non-dimensional) n^{th} harmonic component of the induced velocity. The summation over n is necessary because so many harmonics are present and, as we shall see below, there is some mutual interference between them. For the purposes of the present analysis we must assume that the $\lambda_n(\psi)$ are known from some other source, and that they are not affected by any blade motion which they produce. This is probably a legitimate assumption since, as Miller has pointed out⁽¹⁴⁾, these induced velocity variations are primarily due to the shape and mean strength of the trailing vortex system, parameters which are fixed by performance considerations.

But though the induced velocity is assumed to

cause a variation of incidence around the disc, we shall assume in the theory that shed vorticity is convected away from all points of the disc at the mean induced velocity. This assumption of a 'rigid' wake is certainly not valid, but not much is yet known about the movement of vortices after they have left the blades (for a review of some of the data see (15)). When this behaviour is better understood it will be a straightforward, if complicated, matter to include it in the actuator disc theory. Until that time, however, the rigid wake approximation is probably one of the weakest parts of the present analysis.

The periodic wind speed variation has two effects. The first, which through the lifting line theory is easily absorbed into the actuator disc analysis, is that there are periodic lift changes even if the circulation around a blade is constant with time. The second effect is that because the wind direction is changing, the associated shed vorticity is convected away from the blade in a direction, and at a speed, which depends upon the position of the blade in azimuth. Thus if a blade is on the advancing side shed vorticity is convected away, at right angles to the blade, at a speed which is greater than that due to rotation alone, and if the blade is on the retreating side the convection is reduced. For a blade lying fore and aft the convection velocity normal to the blade is unaffected, but shed vorticity will move parallel to the blade towards the rear, and perhaps out of, the disc. Accounting for this convection provides the main conceptual difficulty of our model.

In forward flight, therefore, conditions are not steady even in a rotating frame of reference, and at a given

instant these conditions vary from blade to blade. It follows that to represent forward flight on the actuator disc model, either these features must be incorporated in some way or they must be shown to be unimportant. Now obviously the variation of relative wind speed with azimuth cannot be ignored, since this is one of the essential differences between forward and hovering flight, but it is probably quite reasonable to neglect its change from blade to blade of the actuator disc model. In fact to be consistent with the assumptions of the hovering theory, this change should be neglected. The reason is that the actuator disc model supposedly represents that part of the flow which is within a few chords length of the reference blade. (It will be remembered that - see Section 2 above - the linear model is allowed to extend to infinity only as an analytical convenience.) At the representative radius a few chords length corresponds to a small arc of azimuth, so that within the compass of the model the variation in relative velocity is small and can be neglected, i.e. it is assumed that the relative wind speed at every point of the actuator disc is the same as it is on the reference blade, but of course this relative speed varies with time. Exactly the same sort of argument was used by Jones⁽²⁾ and Loewy⁽¹⁾ to justify the neglect of the curvature of the wake in the hovering theory. These approximations would not be valid if the tangential velocity varied many times per revolution. Nor, by analogy, is the argument justifying the neglect of wake curvature in hovering valid if the frequency of incidence change is many cycles per revolution, for then the zone of influence of the shed vorticity cannot be considered as limited to a few chords length on either side of the reference blade. But it is clear from the sketch of Figure 2 that the reduction

in induced effects due to wake curvature, and again by analogy, the variation in tangential wind speed over quite a large portion of the disc, depend only on the semi-included angle subtended at the hub by the arc at the representative radius. These reductions and variations will be small for a semi-angle of up to 30° i.e. a total arc length of about one blade radius which is sufficiently large to ensure that all the important induced effects are adequately represented by a linear model.

The general convection of vorticity behind, and to the rear of, the disc means that a blade may never again pass close to a vortex which it shed in an earlier revolution, or that if it does so the shed vortex will probably be no longer parallel to the blade. This point is illustrated in Figure 2. AA represents a vortex shed from a blade in the position OC. Assuming the rotor to be fixed in space and air to be flowing past it at the forward speed V_F the vortex AA will move to the position A'A' in the time it takes OC - or some other blade - to reach the position OB. But A'A' is substantially at right angles to OB so that its influence on OB will be very different from that of AA on OC.

Thus in forward flight the induced effects of a vortex vary with time because the angle which it makes quite different angles with successive blades. So far no method has been devised of including this feature in the two-dimensional actuator disc model, but it is thought that the error in omitting completely will not be too great. The reason is that the induced effect of a particular vortex diminishes with time anyway since the downwash moves it away from the disc. In reality, particularly at low tip

speed ratios, the vortex A'A' would be well below the disc at the instant shown and its absolute contribution to the induced velocity would be small. This particular consideration does, however, suggest that this model of the flow is only likely to be applicable to tip speed ratios μ which are less than, or about equal to, the mean inflow coefficient λ .

In practice the most likely source of error will be in the estimation of the effects of vorticity shed when the blades lie more or less along the line of flight. Since the number of blades is limited, a finite time must elapse before the next blade enters the field of influence of such a shed vortex. During that time the vortex may have moved, parallel to itself, perhaps out of the disc or out of the region of substantially two-dimensional flow, and it may never affect the next blade at all. (Of course, if the blade were really of infinite radius and the flow strictly two-dimensional, moving the vortex parallel to itself would make no difference.) It is possible then that the theory will over-estimate the influence of the shed wake in these regions, but three dimensional and non-rigid wake effects are likely to be of at least equal importance, so there is little point in attempting to refine the theory at this stage.

Since this is a two-dimensional model, spanwise loading variations cannot be included and the effects of wake curvature, distortion and inclination due to forward speed have already been rejected or ignored. The final actuator disc model for forward flight is therefore the same as that for the hovering case except that the relative tangential and induced velocity components are allowed to

vary with time. Thus only these 'three-dimensional' effects which can be expressed as time variations are included, and the best justification for this whole elementary approach is that it is really only meant to apply to the case of vanishingly small μ .

4. 2 Forward Flight Theory

As in the hovering case the governing equations are

$$\frac{\partial^2 \chi}{\partial x^2} + \frac{\partial^2 \chi}{\partial y^2} = -\zeta \quad (60)$$

$$\frac{\partial \zeta}{\partial t} + u_1 \frac{\partial \zeta}{\partial x} + v_1 \frac{\partial \zeta}{\partial y} \quad (61)$$

where $u_1 = \frac{\partial \chi}{\partial y}, v_1 = -\frac{\partial \chi}{\partial x}$ (62)

Once again these equations are linearised, but now we put

$$u_1 = \omega l (1 + \mu \sin \psi) + u \quad (63)$$

$$v_1 = \lambda_0 \omega l + v_1 \quad (64)$$

where $\mu = U_F / \omega l$ is the tip speed ratio and λ_0 is the mean inflow velocity coefficient. Putting equations (60) - (65) in non-dimensional form by means of the transformations (38), equation (60) becomes

$$\frac{\partial^2 F}{\partial \xi^2} + \frac{\partial^2 F}{\partial \eta^2} = -\phi \quad (65)$$

and equation (61) becomes

$$\frac{\partial \phi}{\partial \psi} + (1 + \mu \sin \psi) \frac{\partial \phi}{\partial \xi} + \lambda \frac{\partial \phi}{\partial \eta} = 0 \quad (66)$$

To solve (66) we expand ϕ in ascending powers of μ i.e., we put

$$\phi = \phi_0(\psi, \xi, \eta) + \mu \phi_1(\psi, \xi, \eta) + \mu^2 \phi_2(\psi, \xi, \eta) + \dots \quad (67)$$

This is a common procedure in the treatment of differential equations with periodic coefficients⁽¹⁶⁾, and it has many applications to the helicopter in forward flight (see Jones and Shutler⁽¹⁷⁾).

Substituting (67) in (66) and writing the whole as an expansion in powers of μ we obtain

$$\begin{aligned} & \left(\frac{\partial \phi_0}{\partial \psi} + \frac{\partial \phi_0}{\partial \xi} + \lambda_0 \frac{\partial \phi_0}{\partial \eta} \right) + \mu \left(\frac{\partial \phi_1}{\partial \psi} + \frac{\partial \phi_1}{\partial \xi} + \lambda_0 \frac{\partial \phi_1}{\partial \eta} + \sin \psi \frac{\partial \phi_0}{\partial \xi} \right) \\ & + \mu^2 \left(\frac{\partial \phi_2}{\partial \psi} + \frac{\partial \phi_2}{\partial \xi} + \frac{\partial \phi_2}{\partial \eta} \lambda_0 + \sin \psi \frac{\partial \phi_1}{\partial \xi} \right) + \text{etc.} = 0 \end{aligned} \quad (68)$$

If (68) is to be valid for all values of μ, ψ, ξ, η , then the coefficient of each power of μ must be separately zero. Thus (68) becomes the following set of successive partial differential equations

$$\frac{\partial \phi_0}{\partial \psi} + \frac{\partial \phi_0}{\partial \xi} + \lambda_0 \frac{\partial \phi_0}{\partial \eta} = 0 \quad (69)$$

$$\frac{\partial \phi_1}{\partial \psi} + \frac{\partial \phi_1}{\partial \xi} + \lambda_0 \frac{\partial \phi_1}{\partial \eta} = -\sin \psi \frac{\partial \phi_0}{\partial \xi} \quad (70)$$

plus etc., for higher powers of μ than the first. For the rest of this analysis only the first power of μ will

be retained. This should be sufficient to give the rate of change with μ as μ tends to zero, and the whole approximation does not warrant any further complication.

It will be seen that the equation for ϕ is identical with the equation obtained from the vorticity distribution in the hovering case. The difference is that in forward flight many frequencies will be present in the induced velocity and blade motion and these will, of course, appear in the vorticity distribution.

Hence we put

$$\phi = \sum_n \Phi_{nc} \cos n(\psi - \xi) + \sum_n \Phi_{ns} \sin n(\psi - \xi) \quad (71)$$

and at the same time

$$\begin{aligned} \lambda &= \lambda_0 + \sum_n \lambda_{ns} \sin n\psi + \sum_n \lambda_{nc} \cos n\psi \\ z &= i \sum_n z_{ns} \sin n\psi + i \sum_n z_{nc} \cos n\psi \end{aligned} \quad (72)$$

The substitution (71) makes ϕ independent of η , which is as it must be since the inter-blade phase relationship is such as to make $\omega = \omega_s$ for each frequency.

Substituting (71) in (70) the equation for ϕ becomes

$$\frac{\partial \phi}{\partial \psi} + \frac{\partial \phi}{\partial \xi} + \lambda_0 \frac{\partial \phi}{\partial \eta} = -\sin \psi \left\{ \sum_n n \Phi_{nc} \sin n(\psi - \xi) - \sum_n n \Phi_{ns} \cos n(\psi - \xi) \right\} \quad (73)$$

so that
$$\phi = \cos \psi \left\{ \sum_n n \Phi_{nc} \sin n(\psi - \xi) - \sum_n n \Phi_{ns} \cos n(\psi - \xi) \right\} \quad (74)$$

Thus to a first order in μ

$$\phi = \sum_n \Phi_{ne} \left\{ \cos n(\psi - \xi) + \mu n \cos \psi \sin n(\psi - \xi) \right\} + \sum_n \Phi_{ns} \left\{ \sin n(\psi - \xi) - \mu n \cos \psi \cos n(\psi - \xi) \right\} \quad (75)$$

(75) is now substituted in the partial differential equation (65) to give the non-dimensional stream function. Since (75) is independent of η the complete solution for F_2 is

$$F_2 = \sum_n \frac{\Phi_{ne}}{n^2} \left\{ \cos n(\psi - \xi) + \mu n \cos \psi \sin n(\psi - \xi) \right\} + \sum_n \frac{\Phi_{ns}}{n^2} \left\{ \sin n(\psi - \xi) - \mu n \cos \psi \cos n(\psi - \xi) \right\} + \sum_n C_{2n} e^{n\eta} \cos n(\psi - \xi) + \sum_n S_{2n} e^{n\eta} \sin n(\psi - \xi) \quad (76)$$

where the complementary function has been chosen to ensure finite velocities at infinity.

Since there is no vorticity above the disc

$$F_1 = \sum_n C_{1n} e^{-n\eta} \cos n(\psi - \xi) + \sum_n S_{1n} e^{-n\eta} \sin n(\psi - \xi) \quad (77)$$

The equation of continuity at the actuator disc is

$$\frac{\partial F_{01}}{\partial \xi} = \frac{\partial F_{02}}{\partial \xi} \quad (78)$$

which when evaluated at the reference blade becomes

$$\sum_n \frac{\Phi_{nc}}{n^2} \left\{ n \sin n\psi - \mu n^2 \cos \psi \cos n\psi \right\} + \sum_n \frac{\Phi_{ns}}{n^2} \left\{ -n \cos n\psi - \mu n^2 \cos \psi \sin n\psi \right\} \quad (79)$$

$$+ \sum_n n C_{2n} \sin n\psi - \sum_n n S_{2n} \cos n\psi = \sum_n n C_{1n} \sin n\psi - \sum_n n S_{1n} \cos n\psi$$

Similarly the equation (32) which expresses the condition that changes in circulation around a blade are convected away normal to the disc becomes

$$\sum_n n^2 (C_{1n} + C_{2n}) \sin n\psi + \sum_n n^2 (S_{1n} + S_{2n}) \cos n\psi = \lambda_0 \sum_n \Phi_{nc} \left\{ \sin n\psi + \mu n \cos \psi \sin n\psi \right\} + \lambda_0 \sum_n \Phi_{ns} \left\{ \sin n\psi - \mu n \cos \psi \cos n\psi \right\} \quad (80)$$

Finally the equation (37) connecting the lifting line and actuator disc theories for the non-steady part of the motion becomes

$$\sum_n n (C_{1n} + C_{2n}) \cos n\psi + \sum_n n (S_{1n} + S_{2n}) \sin n\psi = \frac{a}{2} \left\{ \sum_n \left\{ \lambda_{ns} - n z_{nc} - \sigma n C_{1n} \right\} \sin n\psi + \sum_n \left\{ \lambda_{nc} + n z_{ns} + \sigma n S_{1n} \right\} \cos n\psi \right\} \quad (81)$$

In each of equations (79), (80), (81) the various frequencies are collected together, and the coefficients of $\sin n\psi, \cos n\psi$ are equated separately to zero to give six equations for $C_{1n}, S_{1n}, C_{2n}, S_{2n}, \Phi_{nc}, \Phi_{ns}$ in terms of λ_{nc}, z_{ns} , etc. Eliminating the arbitrary constants the following equations are obtained

$$\frac{\Phi_{ns}}{n^2} \left\{ 1 + \frac{4\lambda}{2\sigma} \right\} + \frac{\lambda \Phi_{nc}}{n^2} + \frac{\mu}{2m} \left\{ \Phi_{(m+1)c} + \Phi_{(m-1)c} \right\} = -\frac{2}{n\sigma} \left[\lambda_{nc} + m z_{ns} \right] \quad (82)$$

$$\begin{aligned}
& -\frac{\lambda \Phi_{ms}}{m^2} + \frac{\Phi_{mc}}{m^2} \left\{ 1 + \frac{4\lambda}{2\sigma} \right\} - \frac{\mu}{2m} \left\{ \Phi_{(m-1)s} + \Phi_{(m+1)s} \right\} \\
& = \frac{2}{m\sigma} [\lambda_{ms} - m z_{mc}]
\end{aligned} \tag{83}$$

In arriving at (82), (83) terms in $\mu\lambda$ have been neglected on the grounds that μ and λ are of the same order, and terms in μ^2 have already been ignored as too small.

If we put $\mu=0, \lambda_{mc}=\lambda_{ms}=0$ in (82), (83) the equations reduce to equations (53), (54) for the vorticity components in the hover. To $O(\mu)$ therefore the circulation at a particular frequency ratio m which derives from ϕ since

$$K_t = \lambda \omega l c \int \phi d\psi \tag{84}$$

depends upon the quantities $\Phi_{(m-1)}, \Phi_{(m+1)}$ i.e., upon the circulation at adjacent frequency ratios. This particular harmonic coupling has its origins in the varying convection speed of the vorticity. When the lift is calculated there is an additional inter-harmonic coupling since

$$L = \rho \omega l (1 + \mu \sin \psi) K_t(\psi) \tag{85}$$

Therefore, whereas in hovering it is possible to relate the lift at a given frequency ratio directly to the blade motion at that frequency ratio, in forward flight this cannot be done. Instead it is necessary to carry the calculation one stage further and solve the dynamical equations of motion for the blades, taking into account a whole spectrum of modes and harmonics.

V. DISCUSSION

A comparison of Equations (82) and (83) with equations (53), (54) shows that forward flight affects the shed vorticity - and hence the circulation around the blades - directly through the induced velocity variations, but it also introduces a cross-coupling. Thus, for example, the third harmonic components of circulation around a blade are determined by the third harmonic components of the relative induced velocity, together with some contribution from the second and fourth harmonic circulation components. These inter-harmonic couplings are indeed proportional to μ but a closer inspection of Equations (82) and (83) shows their effects to depend upon the much larger quantity μm . It seems unlikely therefore that the effects of inter-harmonic coupling can be ignored although it may well be that this model of the flow inflates their influence. On the other hand, the inter-harmonic coupling in the lift which arises from the interaction between the fluctuations in relative wind speed and the variations in the circulation - see equation (85) - is only proportional to μ .

The numerical significance of the couplings is not easy to see because the individual circulation components should really be obtained by solving a large number of simultaneous equations. But some special cases and features can be examined and these are most illuminating.

In lifting line theory is employed, the blade airloads at a particular frequency ratio are proportional to the right-hand sides of (82), (83). After solving the equations the constant of proportionality is found to be

the effective lift curve slope or real part of the lift deficiency function. For rotor blades this is usually regarded as being a function only of μ and λ . But if the inter-harmonic coupling is large, equations (82) and (83) show that the constant of proportionality is no longer simple. It must depend upon the circulation content at other harmonics, which in turn depend upon the blade motion and the induced velocity distribution at these harmonics. Our first conclusion then is that, within the limits of velocity of the present theory, it is not legitimate to isolate the lift curve slope from the blade motion and induced velocity. More precisely $(dC_L/d\dot{z})$ can only be defined if the proportion of harmonics in \dot{z} is specified.

The second conclusion we can draw concerns the damping of blade motion. Since Equations (82) and (83) are linear, we can consider the effect of a small disturbance in the blade motion separately. If this is confined to a single harmonic, say the n^{th} so that $\dot{z}_{(n-1)s}, \dot{z}_{(n+1)c}$, etc., are all zero then, because of the cross-coupling, the circulation at the representative section will contain all harmonics. The equations for these additional harmonics will be of the type

$$\frac{\Phi_{(n-1)s}}{m^2} \left\{ 1 + \frac{4\lambda}{2\sigma} \right\} + \frac{\lambda}{m^2} \Phi_{(n-1)c} + \frac{\mu}{2m} \{ \Phi_{(n-2)c} + \Phi_{nc} \} \quad (86)$$

$$- \frac{\lambda \Phi_{(n-1)s}}{m^2} + \frac{\Phi_{(n-1)c}}{m^2} \left\{ 1 + \frac{4\lambda}{2\sigma} \right\} - \frac{\mu}{2m} \{ \Phi_{(n-2)s} + \Phi_{ns} \} \quad (87)$$

The proportion of $\frac{\Phi_{(n-1)c}}{\Phi_{nc}}, \frac{\Phi_{(n-1)s}}{\Phi_{ns}}$ will depend upon μ and the magnitude of the $\Phi_{(n-2)c}, \Phi_{(n-2)s}$ but if the theory is to be valid the $\Phi_{(n-2)c}, \Phi_{(n-2)s}$ etc., must be insignificant. Now suppose the disturbed blade motion to take

place at a frequency ratio μ i.e., the natural frequency of the blade is close to μ cycles per revolution. This means that the generalised inertia force at this frequency almost balances the generalised elastic force, hence the generalised aerodynamic force must be small. Therefore in the present model Φ_m must be small and, from (86), (87) $\Phi_{(m-1)s} \Phi_{(m+1)c}$ will be even smaller. Very roughly

$$\Phi_{(m-1)s} = -\frac{\mu\mu}{2} \Phi_{mc} \quad (88)$$

$$\Phi_{(m-1)c} = \frac{\mu\mu}{2} \Phi_{ms} \quad (89)$$

Substituting (88), (89) and the corresponding expressions for $\Phi_{(m+1)s} \Phi_{(m+1)c}$ into (82), (83) shows that the 'correction' to Φ_m from the interharmonic coupling will be of order $(\mu\mu)^2$ and therefore negligible. Equations (82) and (83) then become identical with the hovering equations (53), (54) which lead directly to the result $C = h/h + \pi$ for the lift deficiency function. We conclude, therefore, that the damping of natural oscillations at a particular frequency is unaffected by μ for small μ .

This result may seem rather surprising, but there is some experimental evidence to support it, and it can be justified theoretically in another way. The experimental evidence is due to Ham, Moser and Zvara⁽¹⁸⁾. They measured the flapping response of the blades of a model helicopter rotor to a vertical excitation of the hub. In the hovering case, for low blade pitch angles, this response showed, in accordance with 'wake' theory, a sharp peak at a frequency of two cycles per revolution (the rotor was of the two-bladed,

hinged type). In forward flight this peak persisted even at $\mu=0.20$. Theoretical justification* can be got by arguing that the flow model and the real rotor system are reversible, i.e., it does not matter whether μ is positive or negative. Hence a plot of the real part of the Theodorsen function against μ , such as that given by Miller⁽¹⁴⁾ must be symmetric about the axis $\mu=0$, i.e. $(\partial \text{Re} / \partial \mu)_{\mu=0} = 0$ and the change with μ can at most be of order μ^2 . This result is sketched in Figure 3, superimposed upon the results of Miller's calculation. Of course the sketch is conjecture - except for the slope at $\mu=0$ - but the horizontal part has been projected to $\mu=0.05$ since the limit of this theory is $\mu \leq \lambda$.

Away from a natural frequency though this result may not hold; there is in fact a difficulty in the concept of damping in such cases. Usually damping is taken to mean the coefficient of a velocity - in this case at a particular frequency ratio or harmonic m . This quantity may perhaps be unchanged by forward motion but an isolated harmonic cannot exist. The inter-harmonic coupling will ensure that blade motion and or vortex shedding will occur at other frequencies. It will be necessary to do work on the system to maintain these motions and manufacture the vorticity. Hence a more appropriate measure of the damping would probably be the energy dissipated in maintaining an oscillation and this will certainly increase with μ .

When we come to consider the more important problem of forced oscillation due to the induced velocity variations, we have to recognise that resonance is difficult

*I am indebted to Dr. Stalle of the Cornell Aeronautical Laboratory for this explanation.

to avoid in practice. Now the generalised aerodynamic force is zero at resonance, and if there is no inter-harmonic coupling the blade incidence must be effectively zero at the resonant frequency. This means that the blade bends in sympathy with the induced velocity variations so that there are no net incidence changes at that frequency. But if the inter-harmonic coupling is strong, then without solving the equations we cannot say what the blade motion will be. The circulation component at the resonant frequency will be small, probably much less than at the adjacent frequencies (since they will not be influenced by resonance). Therefore it is likely that the inter-harmonic coupling terms will determine the blade motion at resonance. This possibility was foreseen by Miller⁽¹⁴⁾.

VI. CONCLUSIONS: FURTHER DEVELOPMENTS

A comparatively crude model of the flow through a rotor has been developed for estimating the influence of the shed wake at low tip-speed ratios. It gives results which are in agreement with more refined theories for the hovering case, and there are reasonable grounds for expecting the model to be valid at least as $\mu \rightarrow 0$. Therefore, although this particular application leaves something to the imagination, it is not felt that any further apology or justification for its use is necessary. A well-established result from hovering theory has been reproduced, and there is adequate physical explanation for many of the theoretical features which are brought out. For instance the parameter λ/σ appears quite clearly as a measure of how much the wake is 'filled' by shed vorticity, the fact that $(\partial \lambda/\partial \mu)_{\mu \rightarrow 0} = 0$ can be deduced from quite separate considerations of symmetry, and even the dependence of the inter-harmonic coupling on the product of tip speed ratio and harmonic number can be explained.

The extension to forward flight proceeds by an expansion in powers of μ , but analysis shows that the expansion really should be in powers of μ^m ; m here is the (integral) order of vibration or frequency ratio. Because of this, it is found that the inter-harmonic couplings, i.e., the generation of aerodynamic forces at one frequency by circulation changes at another, are surprisingly large. Their influence is strong enough to suggest that the simple idea of a lift curve slope independent of the details of the blade motion and induced velocity variation should be abandoned. In particular, the interharmonic couplings

must be taken into account if one or more blade natural frequencies lie close to an integral number of cycles per revolution.

For oscillations at a single frequency at or close to a natural frequency, the effect of interharmonic couplings can be ignored. It is then found that the damping of the oscillations, defined as the aerodynamic coefficient of a blade bending velocity at that frequency, is unaffected by μ . This may account for the apparent persistence of shed wake effects as the tip speed ratio increases.

But care is necessary in interpreting damping in this case, since the oscillation at a single frequency is accompanied by vortex shedding at the coupled frequencies. Work will undoubtedly have to be done on the flow to produce the shed vorticity and this will appear as an energy dissipation somewhere else. In wind tunnel experiments, where model rotors are forced to oscillate by some external agency, there will be three possible sources of energy viz. the tunnel fan motor, the rotor drive and the exciter power supply. Hence it may not be strictly valid to attribute large oscillations to small aerodynamic damping unless measurements of the total power consumption are made.

It remains now to exploit this concept further, either by straightforward application of the actuator disc to other problems or, perhaps better, by an extension of the use of momentum and energy methods. These cannot be expected to yield detailed answers but they do highlight those features which a more elaborate calculation should include. In the spectrum of techniques which are generally

available they lie somewhere between dimensional analysis and numerical solution, and as such they should provide a good guide to future experimental work.

REFERENCES

1. Loewy, R.G., "A Two-Dimensional Approach to the Non-Steady Aerodynamics of Rotary Wings" J. Aerospace Sci. 24 (1957), 82-98
2. Jones, J.P., "The Influence of the Wake on the Flutter and Vibration of Rotor Blades" A.R.C. 18173 January, 1956 (see also Aero. Quarterly, August, 1958)
3. Scheiman, J., "A Tabulation of Helicopter Rotor Blade Differential Pressures, Stresses and Motions as Measured in Flight" NASA TMX-952 March 1954
4. Tararine, S., "Experimental and Theoretical Study of Local Induced Velocities Over a Rotor Disc for Analytical Evaluation of the Primary Loads on Helicopter Rotor Blades" U.S. Army European Research Office Report No. DE2012 October, 1960
- Miller, R.H., "Rotor Blade Harmonic Air Loading" AIAA Journal Vol. 2 No. 7 July, 1964
6. Duwaldt, F., Piziali, R., "Computed Induced Velocity Induced Drag and Angle of Attack Distributions for a Two-Bladed Rotor" Proceedings AHS 19th Annual National Forum, Washington D.C. 1 - 3 May, 1963

7. Willmer, M.A.P., "The Loading of Helicopter Rotor Blades in Forward Flight"
R.A.E. Report (Naval) April, 1959
8. Piziali, R. "Method for Solution of the Aeroelastic Response Problem for Rotating Wings"
Proceedings of the Symposium on the Noise and Loading Actions on Helicopter V/STOL Aircraft and Ground Effect Machines, University of Southampton, August 30 - September 3, 1965
9. Jones, J.P., "Helicopter Vibrations"
Journal of the Royal Aeronautical Soc. Vol. 64, No. 600 December 1960.
10. Sears, W.R., "Rotating Stall in Axial Compressors"
Z.A.M.P. Vol.VI 1955 pp.429-455
11. Whitehead, D.S., "Vibration of Cascade Blades Treated by Actuator Disc Methods"
Proceedings Institute Mechanical Eng. Vol. 173, No.21, 1959
12. Mises, R.V., "Theory of Flight"
Ch.III, McGraw-Hill Book Co. Inc.
13. Young, M.I., "An Engineering Approximation for the Aerodynamic Loading of Rotor Blades"
Proceedings of the Symposium on the Noise and Loading Actions on Helicopter V/STOL Aircraft and Ground Effect Machines

University of Southampton, August 30 -
September 3, 1965

14. Miller, R.H., "Unsteady Airloads on Helicopter Rotor
Blades"
Fourth Cierva Memorial Lecture to the
Royal Aeronautical Soc. Vol. 68 No. 640
April 1964 of the Society's Journal
15. White, R.P., "VTOL Periodic Aerodynamic Loadings"
Proceedings of the Symposium on the
Noise and Loading Actions on Helicopter
V/STOL Aircraft and Ground Effect Machines
University of Southampton, August 30 -
September 3, 1965
16. McLachlan, N.W., "Theory and cation of Mathieu Func-
tions"
Oxford University Press
17. Shutler, A.G., "The Stability of Rotor Blade Flapping
Jones, J.P., Motion"
A.R.C. R & M 3178, 1961
18. Ham, N.D., "Investigation of Rotor Response to
Moser, H.H., Vibrating Aerodynamic Inputs"
Zvara, J., WADC Tech. Report 58-87 Part I
Astia Document No. AD203889 October,
1958

